To descend into the stability domain, the standard ALGOL program for searching for the extremum of a function of five arguments by the gradient method was used. The penalty function (1.10), depending on the Hurwitz inequalities, was minimized by the discrete algorithm in the program.

The lower bound $C_{i\,\min} = 0.0002$, was imposed on the control parameters since they should not be negative. The factor $\Phi$ to accelerate the computations was taken equal to $\Phi = 10^{+15}$ for values of $H_{kk} < +1$.

The descent trajectory of the five control parameters of the automatic system and the stability domain were computed on the BESM-4 computer. The results of the computation are shown in Fig. 2. The values obtained for the control parameters are

$$C_1 = 0.085614, \quad C_2 = 0.005035, \quad C_3 = 0.697956, \quad C_4 = 0.0002, \quad C_5 = 0.000367 \tag{2.5}$$

The penalty function is $M = 5.99939$. For these values of the control parameters the characteristic polynomial is a Hurwitz polynomial; its roots are

$$\lambda_6 = -0.0064, \qquad \lambda_5 = -0.001569 \tag{2.6}$$

$$\lambda_{4,3} = -0.037662 \pm i\, 0.790725, \qquad \lambda_{2,1} = -0.000085 \pm i\, 0.014039$$

BIBLIOGRAPHY

1. Malkin, I. G., Theory of Stability of Motion (Supplement 4 edited by N. N. Krasovskii), Moscow, "Nauka", 1966.

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# ON A THEOREM OF EXISTENCE OF A PERIODIC SOLUTION TO THE LIÉNARD EQUATION

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The criterion of existence of a periodic solution of the Liénard equation

$$x^{\cdot\cdot} + f(x)\, x^{\cdot} + g(x) = 0$$

is established. Definite constraints are imposed on the functions $f(x)$ and $g(x)$, but only for a certain, sufficiently wide range of the values of $x$, containing the coordinate origin.

Let us replace the given equation with an equivalent system given by

$$dx\, /\, dt = y, \qquad dy\, /\, dt = -yf(x) - g(x) \tag{1}$$

and introduce the notation

$$F(x) = \int_0^x f(x)\, dx, \qquad G(x) = \int_0^x g(x)\, dx, \qquad Q(x) = 2G(x) - {}^1/_4\, \lambda^2 x^2 + \lambda \int_0^x F(x)\, dx$$

$$p(x) = 2F(x) - \lambda x, \qquad r(x) = 2G(x) + F^2(x) - \lambda x F(x) + \lambda \int_0^x F(x)\, dx$$

where $\lambda$ is any positive real number for which the conditions of the theorem hold.

Theorem. Let $f(x)$ and $g(x)$ be such that the conditions of the theorem on the existence and uniqueness of the solution hold for the system (1) and

1°. Function $f(0) < 0$.

2°. Numbers $a < b < 0 < c < d$ and $\lambda > 0$ exist such that the functions $g(x)$ and $p(x)$ change their sign in the manner shown below

$$\begin{array}{ll} g(x) < 0 \text{ when } x \in (a, 0), & g(x) > 0 \text{ when } x \in (0, d) \\ p(x) < 0 \text{ when } x \in (a, b), & p(x) > 0 \text{ when } x \in (b, 0) \\ p(x) < 0 \text{ when } x \in (0, c) & p(x) > 0 \text{ when } x \in (c, d) \end{array}$$

3°. For $x \in [b, c]$,

$$M = \min\{Q(a), Q(d)\} > Q(x) + [\sqrt{-\lambda^{-1} p(x) g(x)} + {}^1/_2 |p(x)|]^2$$

Then the system (1) has at least one limit cycle.

Proof. Consider the family of curves

$$\Phi(x, y) = y^2 + p(x)y + r(x) = C \tag{2}$$

Solving it for $y$ and taking into account the notation introduced above, we obtain

$$y = -{}^1/_2\, p(x) \pm \sqrt{C - [r(x) - {}^1/_4\, p^2(x)]} = -{}^1/_2\, p(x) \pm \sqrt{C - Q(x)} \tag{3}$$

Since

$$Q'(x) = 2g(x) - {}^1/_2\lambda^2 x + \lambda F(x) = 2g(x) + {}^1/_2\lambda\,[2F(x) - \lambda x] = 2g(x) + {}^1/_2\lambda p(x)$$

by Condition 2° we have

$$Q'(x) < 0 \quad \text{for } x \in (a, b), \qquad Q'(x) > 0 \quad \text{for } x \in (c, d)$$

i.e. $Q(x)$ decreases monotonously in the interval $(a, b)$ and increases monotonously in the interval $(c, d)$. From $x = b$ and $x = c$ the inequality in Condition 3° yields

$$Q(a) > Q(b), \quad Q(d) > Q(b), \quad Q(a) > Q(c), \quad Q(d) > Q(c)$$

This means that the segments $[Q(b), Q(a)]$ and $[Q(c), Q(d)]$ of the $y$-axis intersect and the number $M$, the smallest of the numbers $Q(a)$ and $Q(d)$, is the upper end of the segment common to both these segments.

Let

$$m = \sup\{Q(x)\} \qquad (b \leqslant x \leqslant c)$$

Obviously $m < M$, otherwise Condition 3° would not hold. From the definition of $m$ and $M$ it follows that the segment $[m, M]$ is contained within the intersection of two segments $[Q(b), Q(a)]$ and $[Q(c), Q(d)]$.

Let $C$ be a fixed number satisfying the inequality $m < C \leqslant M$. Then the equation $Q(x) = C$ has two roots $x_1$ and $x_2$, in this case $a \leqslant x_1 \leqslant b$ and $c \leqslant x_2 \leqslant d$.

Let $x_1 < x_0 < x_2$. Then if $x_0 \in (a, b)$ or $x_0 \in (c, d)$, we have $Q(x_0) < C$ by virtue of the monotonous behavior of $Q(x)$ within these intervals. On the other hand, if $x_0 \in [b, c]$, we have $Q(x_0) < C$ by virtue of the fact that $C > m = \sup\{Q(x)\}$ for $b \leqslant x \leqslant C$.

In any case, according to (3) there are two real values of $y$ on (3) corresponding to each value of $x = x_0$ when $x_1 < x_0 < x_2$, and one value of $y$ corresponding to each of the values $x = x_1$ and $x = x_2$. This means that a simple closed curve belonging to the family (2) and corresponding to the selected value of $C \in (m, M]$ lies between two straight lines $x = x_1$ and $x = x_2$. Since $p(0) = 0$ and $Q(0) = 0$, then by $y(0) = \pm\sqrt{C}$ and this implies that the curves of (2) enclose the coordinate origin.

Equation (3) also implies directly that an increase in the value of $C$ is accompanied by an increase in the distance between the points of intersection of the curves with any straight line parallel to $y$-axis, i. e. if $C_2 > C_1$, the curve $\Phi(x, y) = C_2$ contains within it the curve $\Phi(x, y) = C_1$.

Differentiation of (2) followed by simplification yields, with (1) and the notation introduced above taken into account,

$$d\Phi / dt = -\lambda y^2 - p(x) g(x)$$

We shall show that $d\Phi / dt < 0$ on the curve $\Phi(x, y) = M$.

By 2°, the product $p(x) g(x) > 0$ on the segments $(a, b)$ and $(c, d)$. Consequently $d\Phi / dt < 0$ on those parts of the curve, which lie within two strips, one bounded by the lines $x = a$ and $x = b$, and the other by $x = c$ and $x = d$.

We now find the sign of $d\Phi / dt$ on the upper arc $y_1$ and the lower arc $y_2$ of the curve

$$y_1 = -{}^1/_2\, p(x) + \sqrt{M - Q(x)}, \qquad y_2 = -{}^1/_2\, p(x) - \sqrt{M - Q(x)}$$

in the interval $(b, c)$. Condition 3° gives

$$M - Q(x) > [\sqrt{-\lambda^{-1} p(x) g(x)} + {}^1/_2 \,|p(x)|]^2 \text{ when } x \in [b, c]$$

from which we have

$$-{}^1/_2\,|p(x)| + \sqrt{M - Q(x)} > \sqrt{-\lambda^{-1} p(x) g(x)}$$

Let $b < x < 0$. Then $p(x) > 0$,

$$y_1 = -{}^1/_2\,|p(x)| + \sqrt{M - Q(x)} > \sqrt{-\lambda^{-1} p(x) g(x)}, \qquad y_1^2 > -\lambda^{-1} p(x) g(x)$$

hence

$$d\Phi / dt = -\lambda y_1^2 - p(x) g(x) < 0$$

Since $p(x) > 0$, we have

$${}^1/_2\, p(x) + \sqrt{M - Q(x)} > -{}^1/_2\, p(x) + \sqrt{M - Q(x)}, \quad \text{or} \quad -y_2 > y_1$$

consequently

$$-y_2 > \sqrt{-\lambda^{-1} p(x) g(x)}, \qquad y_2^2 > -\lambda^{-1} p(x) g(x)$$

hence

$$d\Phi / dt = -\lambda y_2^2 - p(x) g(x) < 0$$

In a similar manner we can show that $d\Phi / dt < 0$ on the upper and the lower arc of the curve $\Phi(x, y) = M$ in the interval $(0, C)$. It follows that $d\Phi / dt < 0$ on the curve $\Phi(x, y) = M$ belonging to (2).

Let us consider another family of curves

$$\varphi(x, y) = {}^1/_2\, y^2 + G(x) = C \tag{4}$$

We can easily see that this also represents a family of closed curves enclosing each other, containing the coordinate origin and such, that the value of $C$ increases on the passage from the inner to the outer curves.

Differentiating (4) we obtain, by (1),

$$d\varphi / dt = -y^2 f(x)$$

Since by Condition 1° $f(0) < 0$, $d\varphi / dt \geqslant 0$ on all curves of (4) corresponding to sufficiently small values of $C$. One of these curves and the curve $\Phi(x, y) = M$ together yield an annular region into which all trajectories of system (1) are directed. As it contains no singular points, it must contain at least one stable limit cycle.

BIBLIOGRAPHY

1. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations. Ed. 2, (pp. 152-153), M.-L., Gostekhizdat, 1949.
2. Dragilev, A. V., Periodic solutions of the differential equation of nonlinear vibrations. PMM Vol. 16, №1, 1952.
3. Khudai-Verenov, M. G., Theorems on the limit cycles for the Liénard equations. Usp. matem. n., Vol. 12, №3(75), 1957.

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# A PARTICULAR SOLUTION OF THE PRANDTL-REUSS EQUATION

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The complete dynamic equations of Prandtl-Reuss [1] are examined in the rectangular region. An exact solution is given for a problem which corresponds to some specially selected boundary conditions and initial conditions.

The obtained solution is used to evaluate the correctness of some assumptions which are applicable in the approximate solution of these equations [2].

**1.** The equations of Prandtl-Reuss are used for the description of dynamic processes in such different media as metals and soils. These equations have the form

$$s_{ij}s_{ij} = T(p), \qquad ds_{ij}/dt + \lambda s_{ij} = 2Ge_{ij} \tag{1.1}$$

where

$$s_{ij} = -\sigma_{ij} + p\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - {}^1/_3\varepsilon_{ll}\delta_{ij}, \qquad \lambda = (2Ge_{ij}s_{ij} - {}^1/_2 dT/dt)/T$$

Here $\sigma_{ij}$ and $\varepsilon_{ij}$ are tensors of stresses and velocities of deformation, $p = {}^1/_3\sigma_{ii}$ is the pressure, $G$ is the shear modulus, the operator $d/dt$ is an absolute derivative in the sense of [3]. (It is assumed that the summation is carried out over recurring indices $i, j, k = 1, 2, 3$. Compressive stresses are taken as positive.)

The first of equations (1.1) is the plasticity condition of Mises. The function $T(p)$ which enters into this condition is taken in the form $T = 2(kp + b)^2$ where $k$ and $b$ are constants. The particular form of $T(p)$ was selected by us on the basis of mathematical convenience. However, experimental data for the soil [4] give just this type of relationship.

The remaining equations (1.1) express the condition of coaxiality of stress tensors and velocity tensors of plastic deformations. The value of $\lambda$ is selected such that the condition of plasticity is a consequence of these equations. In this connection it is assumed that $\lambda > 0$. If it turns out that $\lambda < 0$, then (1.1) should be replaced by the conventional equations of elasticity.

The system of equations (1.1) must be closed by means of some relationship between the pressure and the density. This relationship can be quite complex. For example, it can contain hysteresis loops.

So far not a single fairly general solution of equations (1.1) is known. In the solution of specific problems, therefore, these equations are usually simplified. For example, in